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LETTER TO THE EDITOR

The chaotic analytic function

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Abstract. In contrast to a stationary Gaussian random function of a *real* variable which is free to have any correlation function, the closest analogous analytic random function in the *complex* plane has no true freedom—it is (statistically) unique. Since it has arisen only recently, as an apparently universal feature in the physical context of quantum chaos, I refer to it here as 'the chaotic analytic function'. I note that it is implied by the assumption that a quantum chaotic wavefunction has Gaussian randomness and has a constant value for the average of its Wigner function in phase space. Interpreted literally this shows that the chaotic analytic function is the Bargmann function of a pure 'white noise' wavefunction. More physically, if 'constant' is replaced by 'smooth on the scale of a Planck area', these assumptions are the semiclassical ones made by Berry for chaotic eigenstates. The analysis shows that the chaotic analytic function is still obtained semiclassically.

A real stationary Gaussian random function $\psi(x)$ with zero mean, $\langle\psi(x)\rangle = 0$, is fully described by its correlation function $\langle\psi(0)\psi(x)\rangle$. This correlation function, however, is free to have any form (subject to having a positive Fourier transform). The closest analogue of such a random function in the complex plane $\psi(z)$ is, in contrast, greatly restricted by the requirement of analyticity, indeed it has no true freedom at all. It is (statistically) unique in the same sense as 'the Poisson process', or 'the thermal (black body) electromagnetic field' are unique, having no parameters (except size). Since it has arisen only recently, as an apparently universal feature in the physical context of quantum chaos [1–4] I refer to it here as 'the chaotic analytic function'.

The aim, then, is first to derive the chaotic analytic function from the purely mathematical 'closest analogue' standpoint just mentioned. Next, separately, I review the quantum mechanical properties of the Bargmann function [5] of a quantum state. Then I deduce that the chaotic analytic function is formally the Bargmann function of an artificial quantum state whose wavefunction is pure 'white noise' (previous work closely related to this 'white noise' conclusion, by Leboeuf [15] and by Nonnenmacher and Voros [4], is mentioned at the end of (ii) below). This observation alone, however, does not explain why the chaotic analytic function should be universal as the Bargmann function of chaotic quantum states. In (i) I show that under the physically natural assumptions of Berry [6] about the wavefunction of such a state, the chaotic analytic function still results semiclassically.

The uniqueness of the chaotic analytic function is reminiscent of a topic in standard (non-random) complex variable theory. The mere requirement of two-dimensional periodicity in the complex plane leads to a small class of functions—the elliptic functions. The comparison also illustrates another key point. The elliptic functions are not analytic everywhere, they have poles. If a function is to be analytic everywhere ('entire', no singularities except

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at infinity), it is not possible for the function itself to be periodic (unless it is trivially a constant) since its modulus must not have maxima. However its zero points can be, lying on a lattice, and fully characterizing the function by the Weierstrass product theorem. The functions with this property are the theta functions and the elliptic functions are products of ratios of them.

Likewise for the Gaussian random function of a complex variable, if it is to be everywhere analytic, it *cannot* be stationary in the complex plane. However its zero points, which fully characterize it, *can* be a statistically stationary isotropic distribution of points (thus this zero points 'footprint' is really more fundamental than the function itself). This is as close as one can get to 'being stationary' in the complex plane, and is the main defining property of the chaotic analytic function. In both the theta function case and the chaotic analytic function case there is a smooth 'envelope' function $g(z, z^*)$ (non-analytic because of the z^*) for which $g|\psi(z)|^2$ *is* respectively periodic and stationary in the complex plane (see (4) below).

The chaotic analytic function, then, is the (zero mean) analytic Gaussian random function $\psi(z)$ which is statistically isotropic about the origin and has a stationary distribution of zero points in the complex plane. It is fully characterized by its correlation function $\langle\psi(z_1)\psi(z_2)^*\rangle$ which is now derived. The mean density of zero points $\rho(z)$ at z is given at the end of this letter: $\rho(z) = \pi^{-1}\partial^2 \log\langle\psi(z)\psi(z)^*\rangle/\partial z\partial z^*$ [2, 4]. Stationarity requires that ρ is a constant which will be taken as π^{-1} (conventional in the quantum notation later) so $\langle\psi(z)\psi(z)^*\rangle = \psi_0^2 \exp[zz^* + \alpha z + \beta z^*]$, where $\psi_0^2 \equiv \langle\psi(0)\psi(0)^*\rangle$. The statistical isotropy of ψ about the origin requires $\alpha = \beta = 0$, so $\langle\psi(z)\psi(z)^*\rangle = \psi_0^2 \exp[zz^*]$, also $\langle\psi(z)\psi(z)\rangle = 0$. Being analytic, $\psi(z)$ has a Taylor expansion about the origin,

$$\psi(z) = \psi_0 \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

Substituting this into equation $\langle\psi(z)\psi(z)^*\rangle = \psi_0^2 \exp[zz^*]$, implies that the complex coefficients a_n (which are Gaussian distributed since $\psi(z)$ is Gaussian) obey

$$\langle a_n a_m^*\rangle = \delta_{nm}/n! \quad \text{and} \quad \langle a_n a_m\rangle = 0. \tag{2}$$

Finally, using (1) to expand $\psi(z_1)\psi(z_2)^*$ and (2) to average it gives

$$\langle\psi(z_1)\psi(z_2)^*\rangle = \psi_0^2 \exp[z_1 z_2^*] \quad \text{and} \quad \langle\psi(z_1)\psi(z_2)\rangle = 0 \tag{3}$$

fully describing the chaotic analytic function.

When $|\psi(z)|^2$ is multiplied by the envelope function $g(z, z^*) = \psi_0^2/\langle\psi(z)\psi(z)^*\rangle = \exp[-zz^*]$ a positive real isotropic stationary random function in the complex plane is produced (figure 1):

$$H(z) \equiv \exp[-zz^*]|\psi(z)|^2. \tag{4}$$

(This is not Gaussian because of the power two, and not analytic because of the complex conjugates.) The correlation function of H is easily found, by using the rule (10) for Gaussian random functions:

$$\langle H(z_1)H(z_2)\rangle = \psi_0^4[1 + \mathrm{e}^{-|z_1-z_2|^2}]. \tag{5}$$

In the context of quantum mechanics, setting aside randomness, let $\psi(z)$ denote the 'Bargmann function' of the quantum state $|\psi\rangle$, whose relevant properties I now review. In terms of the position representation $\psi(q)(\equiv \langle q|\psi\rangle)$, taking $\hbar = 1$ here and henceforth,

$$\psi(z) = \frac{1}{\pi^{1/4}} \mathrm{e}^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} \psi(q) \mathrm{e}^{-\frac{1}{2}(q-\sqrt{2}z)^2}\, \mathrm{d}q. \tag{6}$$

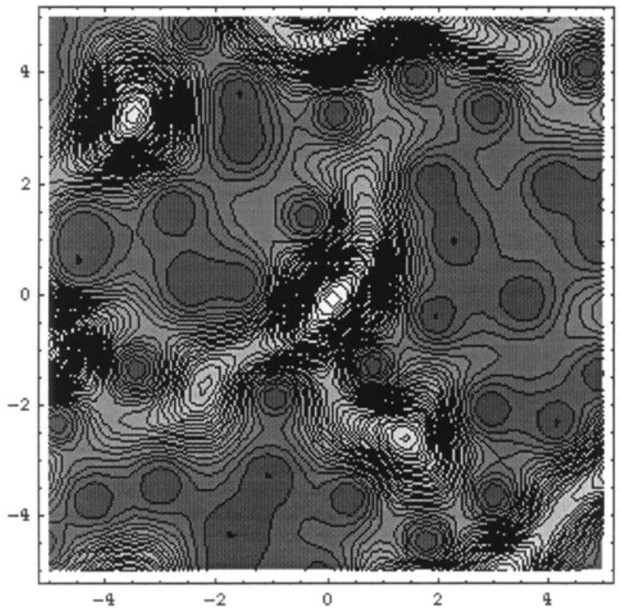

Figure 1. Contour plot of $H(z) = |\psi(z)|^2 \exp(-zz^*)$ in the complex plane, where $\psi(z)$ is the chaotic analytic function. $H(z)$ is an isotropic stationary random function. Low values are dark (contour crowding excepted).

Thus, apart from a multiplying factor and a scaling factor $\sqrt{2}$, the Bargmann function is an analytic continuation into the complex plane of the convolution of $\psi(q)$ with a Gaussian. More physically it is (a factor times) the inner product of $|\psi\rangle$ with a 'coherent state' associated with $z = x + iy$: a ground state of the harmonic oscillator with Hamiltonian $\frac{1}{2}(p^2 + q^2)$ displaced in position by $\sqrt{2}x$, and in momentum by $-\sqrt{2}y$.

As a quantum object associated with phase space (q, p), the Bargmann function $\psi(z)$, was recognized by Voros [7] as a tool for semiclassical mechanics and quantum chaos [1, 3, 4, 8–13]. It complements the other well known quantum phase space object—the Wigner function $W(q, p)$ which is a real function in phase space

$$W(q, p) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi\left(q + \frac{1}{2}Q\right) \psi\left(q - \frac{1}{2}Q\right)^* \mathrm{e}^{-\mathrm{i}pQ}\, \mathrm{d}Q. \tag{7}$$

The two functions, $\psi(z)$ and $W(q, p)$, are related via the 'Husimi function' [14] in phase space defined as the convolution of W with the function $(\pi)^{-1} \exp(-q^2 - p^2)$:

$$\frac{1}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(q', p')\mathrm{e}^{-(q-q')^2-(p-p')^2}\, \mathrm{d}q'\, \mathrm{d}p'. \tag{8}$$

Remarkably, as a standard algebraic manipulation shows, this Husimi function is just that function H defined in (4), with $\sqrt{2}z = q - ip$. This shows that the Husimi function is everywhere positive (unlike the Wigner function). One final property of the Bargmann function $\psi(z)$ will be useful: for the quantum state $|n\rangle$ which is the nth excited energy eigenstate of the harmonic oscillator $\frac{1}{2}(q^2 + p^2)$, the Bargmann function is

$$\psi_{(n)}(z) = \frac{z^n}{\sqrt{n!}}. \tag{9}$$

Having derived the chaotic analytic function as the closest analogue of a stationary Gaussian random function and recalled the required features of the quantum Bargmann function I move on now to the connection between them.

Consider any (zero mean) Gaussian random wavefunction $\psi(q)$, stationary or otherwise, but whose overall phase is uniformly random. It is fully described by its correlation function $\langle\psi(q_1)\psi(q_2)^*\rangle$ (since the unconjugated pair $\langle\psi(q_1)\psi(q_2)\rangle$ is zero by phase randomness). Higher moments are given by the pair product decomposition rule for Gaussian random functions. For example, using subscripts for brevity,

$$\langle\psi_1\psi_2^*\psi_3\psi_4^*\rangle = \langle\psi_1\psi_2^*\rangle\langle\psi_3\psi_4^*\rangle + \langle\psi_1\psi_4^*\rangle\langle\psi_3\psi_2^*\rangle + \langle\psi_1\psi_3\rangle\langle\psi_2^*\psi_4^*\rangle \tag{10}$$

where the final term is zero. This is to be used to obtain first the correlation function for the Wigner function, and then that of the Husimi function for comparison with the chaotic analytic result (5).

Taking the average of the definition (7) of the Wigner function $W(q,p)$ shows that the phase space function $\langle W(q,p)\rangle$, is the Fourier transform of the correlation function of ψ, and therefore fully describes its statistics. In particular the mere average $\langle W(q,p)\rangle$ fully determines all the higher correlation statistics of W. The correlation function $\langle W_1W_2\rangle$ (meaning $\langle W(q_1,p_1)W(q_2,p_2)\rangle$) of W involves the product of four ψ which reduces by (10) to two terms. After a little algebra one obtains

$$\langle W_1W_2\rangle = \langle W_1\rangle\langle W_2\rangle + \frac{2}{\pi}\iint\langle W_{1'}\rangle\langle W_{2'}\rangle \mathrm{e}^{\mathrm{i}[(1-2)\wedge(1'-2')]}\delta(1+2-1'-2')\,\mathrm{d}^21'\,\mathrm{d}^22'. \tag{11}$$

Thus, contributing to the correlation at points 1 and 2 in phase space, are the values of $\langle W\rangle$ at all pairs of points $1'$ and $2'$ with the same midpoint as 1 and 2; they form a parallelogram $12'21'$. They contribute with a phase equal to the cross product of the diagonal vectors or twice the parallelogram area.

The correlation function for the Husimi function (the Wigner function blurred with the Gaussian (8)) is obtained directly from (11)

$$\langle H_1H_2\rangle = \frac{1}{\pi^2}\iint\langle W_1W_2\rangle \mathrm{e}^{-(1-1')^2}\mathrm{e}^{-(2-2')^2}\,\mathrm{d}^21'\,\mathrm{d}^22' \tag{12}$$

$$= \langle H_1\rangle\langle H_2\rangle + \frac{2}{\pi}\iiiint\langle W_{1''}\rangle\langle W_{2''}\rangle \mathrm{e}^{\mathrm{i}(1'-2')\wedge(1''-2'')}\mathrm{e}^{-(1-1')^2}\mathrm{e}^{-(2-2')^2}$$
$$\times\delta(1'+2'-1''-2'')\mathrm{d}^21'\,\mathrm{d}^22'\,\mathrm{d}^21''\,\mathrm{d}^22'' \tag{13}$$

$$= \langle H_1\rangle\langle H_2\rangle + \frac{1}{\pi^2}\iint\langle W_{1''}\rangle\langle W_{2''}\rangle \mathrm{e}^{\mathrm{i}(1-2)\wedge(1''-2'')}\mathrm{e}^{-\frac{1}{2}(1''-2'')^2}$$
$$\times\mathrm{e}^{-\frac{1}{2}((1+2-1''-2'')^2}\,\mathrm{d}^21''\,\mathrm{d}^22''. \tag{14}$$

The connection with the pure chaotic analytic function is now straightforward; if the average Wigner function $\langle W(q,p)\rangle$ is a constant throughout phase space, then $\langle H\rangle = \langle W\rangle$ and (14) reduces to the Husimi correlation (5) for the chaotic analytic function (with $\sqrt{2}z = q - \mathrm{i}p$). This is enough; it means that $|\langle\psi(z_1)\psi(z_2)^*\rangle| = \psi_0^2|\exp(z_1z_2^*)|$, and the modulus of an analytic function fixes the function itself (up to overall phase) so the modulus signs can be removed. Thus, the assumption that the wavefunction $\psi(q)$ is Gaussian random and that its associated average Wigner function is constant leads, exactly, to the Bargmann function for the quantum state being the chaotic analytic function. By taking the average of (7) it follows that the correlation function for the wavefunction is the Fourier transform of a constant, namely a delta function. Thus, the chaotic analytic function is the Bargmann function of pure Gaussian ‘white noise’, $\langle\psi(q_1)\psi(q_2)^*\rangle = \delta(q_1 - q_2)$. There follow several remarks on this result.

(i) Of course an everywhere constant $\langle W(q, p)\rangle$ is quite unphysical and correspondingly the $\psi(z)$ for a physical system could not be the chaotic analytic function throughout the whole complex plane. Normally, W is zero for $p \to \infty$, for instance. One might be tempted to argue, though, that the wavefunction $\psi(q)$ of a chaotic state simulates white noise in the semiclassical limit and that this therefore suffices to explain why $\psi(z)$ is, apparently universally, the chaotic analytic function. This would be unjustified, however, because the form of W (or $\psi(z)$) in the remote regions (large p, for instance) probes exactly the fine structure of the simulated white noise $\psi(q)$—it probes its 'colour'. To be consistent with the apparent universality of the chaotic analytic function the association between $\langle W\rangle$ and $\psi(z)$ (or rather the correlation statistics of $\psi(z)$) should be a *local* one in phase space. To verify this I examine first the form of the Wigner correlation function.

From the comments following (11) it is clear that the Wigner correlation function is influenced by the behaviour of $\langle W\rangle$ at arbitrarily remote points as well as near ones. For example, suppose $\langle W(q, p)\rangle = 1$ in the disc $q^2 + p^2 < r^2$, and zero outside, which is approximately realized by truncating the series (1) (so that, in the language of equation (9), high oscillator states are eliminated). Take a simple circumstance in which the midpoint between points 1 and 2 lies at the origin. The formula (11) then reduces to the two-dimensional Fourier transform of a disc which is the Bessel function $J_0(2r|1-2|)$. The larger the circle radius r, the smaller the scale of correlation function. The local behaviour of the Wigner correlation function depends intimately on the global behaviour of its average $\langle W\rangle$. The correlation statistics of W are not universal.

In contrast the correlation statistics of the Husimi function H are universal. This follows by examining the integral for the Husimi correlation function (14). The two real exponents add to give $-[1'' - \frac{1}{2}(1+2)]^2 - [2'' - \frac{1}{2}(1+2)]^2$ so that the integral only strongly samples points $1''$ and $2''$ in the neighbourhood of the midpoint of 1 and 2. Therefore only the local behaviour of $\langle W\rangle$ matters: further away from this midpoint than a distance of order unity the value of $\langle W\rangle$ is irrelevant to the integral. If the scale over which $\langle W\rangle$ varies is much larger than this, as it is semiclassically since the Planck scale is unity by construction, $\langle W\rangle$ can be treated as a constant. In this case the Husimi correlation function is once again given by (5) and the local Bargmann function $\psi(z)$ is guaranteed to be chaotic analytic.

This, then, is the desired conclusion. The two assumptions, that the wavefunction $\psi(q)$ is Gaussian random, and that the average Wigner function is smooth on the scale of a Planck area (i.e. locally constant) were just the semiclassical ones made by Berry [6] for chaotic quantum states. More accurately he took the local average Wigner function to be proportional to the singular function $\delta(H(\boldsymbol{q}, \boldsymbol{p}) - E)$ in $2N$-dimensional phase space. This requires reduction to a two-dimensional phase space by section/projection in order to obtain a Bargmann function. The reduction produces the smooth average Wigner function required.

Further comments on the above conclusion that the Bargmann function $\psi(z)$ is semiclassically the chaotic analytic function may be in order. Its intensity, as measured by $\langle|\psi(z)|^2\rangle \exp(-zz^*)(\equiv H \approx \langle W\rangle)$, will vary classically in phase space. But everywhere, locally, on a scale which is classically small but much larger than the Planck area ($\sim$1), the Bargmann function will be the chaotic analytic function with the standard correlation scale (mean density of zero points π^{-1} in the complex plane, or $(2\pi)^{-1}$ in the phase space plane). Any deviations from the chaotic analytic function vanish semiclassically. Small though such deviations must be, they have important consequences. For example as pointed out by Nonnenmacher and Voros [4] the density deviation $\rho - \pi^{-1}$ of zero points equals $\nabla^2 \log(H)$, so that an exponentially small deviation can cause a substantial change in the intensity $\langle H\rangle$ over a global scale. Conversely since $\langle H\rangle$ usually does vary globally there must exist semiclassically small density deviations.

(ii) The white noise wavefunction result can be anticipated by another argument which it is worth outlining briefly. Directly from (1), (2) and (9) the chaotic analytic function is the Bargmann function of the quantum state which is a superposition of all the energy eigenstates $|n\rangle$ of a harmonic oscillator $\frac{1}{2}(p^2+q^2)$ with *equally* distributed independent Gaussian random complex coefficients b_n. This superposition has $\Sigma\langle\langle q_2|n\rangle b_n b_n^*\langle n|q_1\rangle\rangle = \delta(q_1-q_2)$ (because the average of the central operator is the identity operator) and the white noise result is demonstrated. Moreover, the choice of this harmonic oscillator basis is seen to be quite unnecessary—any complete orthonormal basis $|n\rangle$ (on the full q axis) can be superposed with equally distributed independent complex Gaussian random coefficents to obtain white noise and hence the chaotic analytic function. Another way to say this is that the chaotic analytic function is invariant under any unitary transformation (i.e. unitary change of basis).

The invariance property answers an awkward question that arises when using (6) to construct the Bargmann function $\psi(z)$ for a chaotic quantum state $\psi(q)$. The system will generally have no natural choice of harmonic oscillator Hamiltonian—why should one choose the 'unit frequency' case, $\frac{1}{2}(p^2+q^2)$ of (6)? One need not: if the $\psi(z)$ is going to be the chaotic analytic function as it ought, any choice will do.

To conclude this remark I summarize the relevant parts of previous work by others in the same direction. The invariance property has been mentioned before by Leboeuf [15] with reference to the eigenstate of a random matrix which is a model for a chaotic wavefunction. He notes that these have Gaussian randomness in common with those of the Berry assumptions, but contrasts the correlation calculations in real space and the complex plane. Nonnenmacher and Voros [4] consider a 'statistical model' of a wavefunction as a Fourier series of a finite number N of terms with independent complex Gaussian random coefficients. In the limit $N\to\infty$ (which would correspond to a 'periodic' white noise) they show that the Bargmann function approaches the chaotic analytic function (periodized, of course, but on a scale of many correlation lengths).

(iii) A special, but physically important, case arises if the chaotic wavefunction $\psi(q)$ is real. Then $\psi(z)$ approximates the chaotic analytic function except near the real axis (within a distance of order unity). The Wigner and Husimi correlation functions, (11) and (14), have an extra integral, arising from the final term of (10), in which the vector $2\equiv(q_2,p_2)$ is replaced by its reflection in the q axis $2^*\equiv(q_2,-p_2)$. (The Wigner and Husimi functions are symmetric under this reflection for real $\psi(q)$.) Accordingly, in the local constancy approximation for $\langle W\rangle$, the Husimi correlation becomes $\psi_0^4(1+\exp-|z_1-z_2|^2+\exp-|z_1-z_2^*|^2)$ instead of (5). The distribution of zero points of $\psi(z)$ in the case of the real $\psi(q)$ case was found by Prosen [16], differing from that of the chaotic analytic function near the real axis. Prosen, incidentally, also found the parameter dependence of the zero point statistics [17] for both the complex and real cases.

Finally the derivation [2, 18, 19] of the density of the zero points of a complex Gaussian random function is recalled. In terms of the joint probability $P(\psi,\psi')$ of the value of the function ψ and its derivative $\psi'\equiv \mathrm{d}\psi/\mathrm{d}z$ at some point z, the probability of a zero point in the region d^2z is $\rho\,\mathrm{d}^2z$ where $\rho=\int P(0,\psi')|\psi'|^2\,\mathrm{d}^2\psi'$. But $P(\psi,\psi')=\pi^{-2}\det M^{-1}\exp[-(\psi,\psi'^*)M^{-1}(\psi,\psi'^*)]$ where M is the 2×2 matrix

$$M=\begin{bmatrix}\langle\psi\psi^*\rangle & \langle\psi\psi'^*\rangle\\ \langle\psi'\psi^*\rangle & \langle\psi'\psi'^*\rangle\end{bmatrix}\equiv\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}. \tag{15}$$

So

$$\rho=\int P(0,\psi')|\psi'|^2\,\mathrm{d}^2\psi' = \text{coeff of } \mu\mu^* \text{ in } -\int P(0,\psi')\exp[\mathrm{i}(\mu^*\psi'+\mu\psi'^*)]\,\mathrm{d}^2\psi'. \tag{16}$$

The integrals are Gaussian and yield

$$\rho = [C - B^* A^{-1} B]/\pi A = \pi^{-1} \partial^2 \log\langle \psi(z)\psi(z)^* \rangle / \partial z \partial z^* \qquad (17)$$

where the compact second form here was observed in [4]. This mean density is all that was required, but I pursue the account briefly to describe the correlation or joint probability density $\rho(z_1, z_2)$ of zero points at z_1 and z_2 which gives an indication of the statistical arrangement of the points [2]. Generalizing (16) with A, B, and C themselves representing 2×2 matrices, $A_{ij} \equiv \langle \psi_i \psi_j^* \rangle$ etc, gives the required generalization of (17), $\rho(z_1, z_2) = \mathrm{per}[C - B^\dagger A^{-1} B]/\pi^2 \det A$, where per is the 'permanent' of the matrix. Substituting (3) one obtains $\rho(z_1, z_2) = g(|z_1 - z_2|/\sqrt{2})$ where

$$g(r) = [(\sinh^2 r^2 + r^4)\cosh r^2 - 2r^2 \sinh r^2)]/\sinh^3 r^2. \qquad (18)$$

(This is equation (12) of [2]. In the preceding line of this reference the quoted argument of g is too large by a factor $\sqrt{2}$. The line is unfortunately repeated in the abstract but affects nothing else.) The function $g(r)$ is quadratic at the origin indicating 'repulsion' of the points and has a single small hump before decaying exponentially to unity at infinity.

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